

SQUARE ROOTS OF HAMILTONIAN Diffeomorphisms

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ABSTRACT. In this article we prove that on any closed symplectic manifold there exists an arbitrarily C^∞ -small Hamiltonian diffeomorphism not admitting a square root.

1. INTRODUCTION

Let (M, ω) be a closed symplectic manifold, i.e. $\omega \in \Omega^2(M)$ is a non-degenerate, closed 2-form. To a function $L : S^1 \times M \rightarrow \mathbb{R}$ we associate the Hamiltonian vector field X_L by setting

$$\omega(X_{L_t}, \cdot) = -dL_t(\cdot) \quad (1)$$

where $L_t(x) := L(t, x)$. The flow $\phi_L^t : M \rightarrow M$ of the vector field X_{L_t} is called a Hamiltonian flow. For simplicity we abbreviate

$$\phi_L = \phi_L^1. \quad (2)$$

The Hamiltonian diffeomorphisms form the Lie group $\text{Ham}(M, \omega)$ with Lie algebra being the smooth functions modulo constants. We refer the reader to the book [MS98] for the basics in symplectic geometry.

In this article we prove the following Theorem.

Theorem 1. *In any C^∞ -neighborhood of the identity in $\text{Ham}(M, \omega)$ there exists a Hamiltonian diffeomorphism ϕ which has no square root, i.e. for all Hamiltonian diffeomorphism ψ (not necessarily close to the identity)*

$$\psi^2 \neq \phi \quad (3)$$

holds.

An immediate corollary of Theorem 1 is the following.

Corollary 2. *The exponential map*

$$\begin{aligned} \text{Exp} : C^\infty(M, \mathbb{R})/\mathbb{R} &\rightarrow \text{Ham}(M, \omega) \\ [L] &\mapsto \phi_L \end{aligned} \quad (4)$$

is not a local diffeomorphism.

In the proof of the Theorem we use the following beautiful observation by Milnor [Mil84, Warning 1.6]. Milnor observed that an obstruction to the existence of a square root is an odd number of $2k$ -cycles, see next section for details. The main work in this article is to construct an example in the symplectic category.

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2. MILNOR'S OBSERVATION

We define

$$CM^k := M^k / (\mathbb{Z}/k) \quad (5)$$

where \mathbb{Z}/k acts by cyclic shifts on M^k . We write elements of CM^k as

$$[x_1, \dots, x_k] \in CM^k. \quad (6)$$

The space of k -cycles of a diffeomorphism $\phi : M \rightarrow M$ is

$$\mathcal{C}^k(\phi) := \{[x_1, \dots, x_k] \in CM^k \mid \phi^j(x_i) \neq x_i \forall j = 1, \dots, k-1, \phi(x_i) = x_{i+1}\}. \quad (7)$$

We point out that if $[x_1, \dots, x_k] \in \mathcal{C}^k(\phi)$ then $\phi^k(x_i) = x_i$ for $i = 1, \dots, k$.

Proposition 3 (Milnor [Mil84]). *If $\phi = \psi^2$ then $\mathcal{C}^{2k}(\phi)$ admits a free $\mathbb{Z}/2$ -action. In particular, $\#\mathcal{C}^{2k}(\phi)$ is even if $\mathcal{C}^{2k}(\phi)$ is a finite set.*

For the convenience of the reader we include a proof of Milnor's ingenious observation.

PROOF. We define

$$\begin{aligned} I : \mathcal{C}^{2k}(\phi) &\rightarrow \mathcal{C}^{2k}(\phi) \\ [x_1, \dots, x_{2k}] &\mapsto [\psi(x_1), \dots, \psi(x_{2k})]. \end{aligned} \quad (8)$$

Since $\psi \circ \phi = \phi \circ \psi$ and $\psi^2 = \phi$ the map I is well-defined and an involution. We assume by contradiction that $[x_1, \dots, x_{2k}]$ is a fixed point of I , i.e. there exists $0 \leq r \leq 2k-1$

$$\psi(x_i) = x_{i+r} \quad (9)$$

where we read indices $\mathbb{Z}/2k$ -cyclically. Using $x_{i+r} = \phi^r(x_i)$ we get

$$\psi(x_i) = \phi^r(x_i) = \psi^{2r}(x_i) \quad (10)$$

and thus

$$\psi^{2r-1}(x_i) = x_i. \quad (11)$$

In particular,

$$x_i = \psi^{2r-1}(x_i) = \psi^{2r-1}(\psi^{2r-1}(x_i)) = \psi^{4r-2}(x_i) = \phi^{2r-1}(x_i). \quad (12)$$

In summary we have

$$\begin{aligned} x_i &= \phi^{2r-1}(x_i) \\ x_i &= \phi^{2k}(x_i). \end{aligned} \quad (13)$$

In general, if

$$\begin{aligned} z &= \phi^a(x) \\ z &= \phi^b(x) \end{aligned} \quad (14)$$

for $a, b \in \mathbb{Z}$ then

$$z = \phi^{\text{lcd}(a,b)}(x) \quad (15)$$

since by the Euclidean algorithm there exists $n_1, n_2 \in \mathbb{Z}$ with

$$\text{lcd}(a, b) = n_1 a + n_2 b. \quad (16)$$

In our specific situation $2r-1$ is odd and $2k$ is even and thus

$$1 \leq \text{lcd}(2r-1, 2k) < 2k \quad (17)$$

contradicting the assumption $\phi^j(x_i) \neq x_i \forall j = 1, \dots, 2k-1$. This proves the Proposition. \square

3. PROOF OF THEOREM 1

Let (M, ω) be a closed symplectic manifold. We fix a Darboux chart $B^{2N}(R) \cong B \subset M$ where $B^{2N}(R)$ is the open ball of radius R in \mathbb{R}^{2N} . For an integer $k \geq 1$ and a positive number $\delta > 0$ we choose a smooth function $\rho : [0, R^2] \rightarrow \mathbb{R}$ satisfying the following

$$\begin{aligned} \frac{\pi}{2k} &\geq \rho'(r) > 0, \\ \rho'(r) &= \frac{\pi}{2k} \iff r = \frac{1}{2}R^2, \\ \rho'|_{[\frac{8}{9}R^2, R^2]} &= \delta > 0. \end{aligned} \tag{18}$$

We set

$$\begin{aligned} H : B^{2N}(R) &\rightarrow \mathbb{R} \\ z &\mapsto \rho \left(\sum_{\nu=1}^N \frac{\nu}{N} |z_\nu|^2 \right). \end{aligned} \tag{19}$$

We denote by $\phi_H^t : B^{2N}(R) \rightarrow B^{2N}(R)$ the induced Hamiltonian flow. We recall that the Hamiltonian flow of $z \mapsto |z|^2$ is given by $z \mapsto \exp(2it)z$ thus

$$(\phi_H^t(z))_\nu = \exp \left[\rho' \left(\sum_{\nu=1}^N \frac{\nu}{N} |z_\nu|^2 \right) \frac{2it\nu}{N} \right] z_\nu. \tag{20}$$

We point out that ϕ_H^t preserves the quantities $|z_\nu|$, $\nu = 1, \dots, N$.

Lemma 4. *The fixed points of ϕ_H^{2k} are precisely $z = 0$ and the circle*

$$C := \left\{ (z_1, \dots, z_N) \in B^{2N}(R) \mid |z_N|^2 = \frac{1}{2}R^2 \text{ and } z_1 = \dots = z_{N-1} = 0 \right\}. \tag{21}$$

Moreover, ϕ_H acts on C by rotation of the last coordinate by an angle of $\frac{\pi}{k}$.

PROOF. Assume $\phi_H^{2k}(z) = z$ which is equivalent to

$$\exp \left[\rho' \left(\sum_{\nu=1}^N \frac{\nu}{N} |z_\nu|^2 \right) \frac{2it\nu}{N} \right] z_\nu = z_\nu, \quad \nu = 1, \dots, N, \tag{22}$$

thus, either $z_\nu = 0$ or

$$\rho' \left(\sum_{\nu=1}^N \frac{\nu}{N} |z_\nu|^2 \right) \frac{4k\nu}{N} \in 2\pi\mathbb{Z}. \tag{23}$$

From $\rho'(r) \leq \frac{\pi}{2k}$ we conclude that $z_1 = \dots = z_{N-1} = 0$. Moreover, $z_N = 0$ or

$$\rho' \left(\sum_{\nu=1}^N \frac{\nu}{N} |z_\nu|^2 \right) = \rho'(|z_N|^2) = \frac{\pi}{2k} \tag{24}$$

holds. In summary, either $z = 0$ or $z \in C$. This together with (20) proves the Lemma. \square

We now perturb H . For this we fix a smooth cut-off function $\beta : [0, R^2] \rightarrow [0, 1]$ satisfying

$$\begin{aligned} \beta|_{[\frac{1}{3}R^2, \frac{2}{3}R^2]} &= 1 \\ \beta|_{[0, \frac{1}{9}R^2] \cup [\frac{8}{9}R^2, R^2]} &= 0 \end{aligned} \tag{25}$$

and set

$$F(z) := \beta(|z_N|^2) \cdot \operatorname{Re} \left(\frac{z_N^k}{|z_N|^k} \right) : B^{2N}(R) \rightarrow \mathbb{R} \quad (26)$$

where Re is the real part. If we introduce new coordinates $(z_1, \dots, z_{N-1}, r, \vartheta)$, where $z_N = r \exp(i\vartheta)$, the function F equals

$$F(z) = \beta(r^2) \cos(k\vartheta) . \quad (27)$$

We point out that the Hamiltonian diffeomorphism $\phi_H \circ \phi_{\epsilon F}$ maps $B^{2N}(R)$ into itself.

Lemma 5. *There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$*

$$\#\mathcal{C}^{2k}(\phi_H \circ \phi_{\epsilon F}) = 1 . \quad (28)$$

PROOF. We set

$$D := \left\{ (z_1, \dots, z_{N-1}, r, \vartheta) \in C \mid \vartheta = \frac{j\pi}{k}, j = 0, \dots, 2k-1 \right\} \quad (29)$$

where C is defined in Lemma 4. The same lemma implies that ϕ_H acts on D as a cyclic permutation sending $\frac{j\pi}{k}$ to $\frac{(j+1)\pi}{k}$. Moreover, we have

$$\phi_{\epsilon F} z = z \quad (30)$$

for $z \in D$ since $D \subset \operatorname{Crit} F$. In particular, D corresponds precisely to a single element in $\mathcal{C}^{2k}(\phi_H \circ \phi_{\epsilon F})$. It remains to show that there are no other $2k$ -cycles. We prove something stronger, namely that for sufficiently small $\epsilon > 0$ the only other fixed point of $(\phi_H \circ \phi_{\epsilon F})^{2k}$ is $z = 0$.

For $0 < a < b$ we set

$$A(a, b) := \{ (z_1, \dots, z_{N-1}, r, \vartheta) \in B^{2N}(R) \mid r \in [aR^2, bR^2] \} . \quad (31)$$

We observe that on $A(\frac{1}{3}, \frac{2}{3})$ we have $\beta = 1$ and thus the flow of ϵF is given by

$$(z_1, \dots, z_{N-1}, r, \vartheta) \mapsto (z_1, \dots, z_{N-1}, r - t\epsilon k \sin(k\vartheta), \vartheta) . \quad (32)$$

In particular, if we set

$$\bar{\epsilon} := \frac{R^2}{18k^2} \quad (33)$$

then for $0 < \epsilon < \bar{\epsilon}$ we conclude that

$$(\phi_H \circ \phi_{\epsilon F})^{2k} \left(A\left(\frac{4}{9}, \frac{5}{9}\right) \right) \subset A\left(\frac{1}{3}, \frac{2}{3}\right) , \quad (34)$$

since ϕ_H^t preserves the r coordinate. Fix $w \in A(\frac{4}{9}, \frac{5}{9})$ with $(\phi_H \circ \phi_{\epsilon F})^{2k}(w) = w$ and set for $j = 0, \dots, 2k$

$$\begin{aligned} z_\nu^j &:= P_{z_\nu} \left((\phi_H \circ \phi_{\epsilon F})^j(w) \right) , \quad \nu = 1, \dots, N-1 \\ r^j &:= P_r \left((\phi_H \circ \phi_{\epsilon F})^j(w) \right) \\ \vartheta^j &:= P_\vartheta \left((\phi_H \circ \phi_{\epsilon F})^j(w) \right) \end{aligned} \quad (35)$$

where P_{z_ν} , P_r , and P_ϑ are the projections on the respective coordinates. It follows from equation (32) that

$$P_{z_\nu} \left((\phi_H \circ \phi_{\epsilon F})^j(w) \right) = P_{z_\nu}(\phi_H^j(w)) \quad \nu = 1, \dots, N-1 . \quad (36)$$

By the same argument as in the proof of Lemma 4 we conclude

$$z_\nu^j = 0 \quad \forall \nu = 1, \dots, N-1 \text{ and } \forall j = 0, \dots, 2k. \quad (37)$$

Next, it follows from the flow equations (20) and (32)

$$0 < \vartheta_{j+1} - \vartheta_j \leq \frac{\pi}{k} \pmod{2\pi}. \quad (38)$$

By (18) equality holds if and only if $r_{j+1} = \frac{1}{2}R^2$. Using again $(\phi_H \circ \phi_{\epsilon F})^{2k}(w) = w$ we deduce

$$\vartheta_{2k} - \vartheta_0 = 0 \pmod{2\pi} \quad (39)$$

and therefore

$$r_0 = r_1 = \dots = r_{2k} = \frac{1}{2}R^2. \quad (40)$$

In summary

$$w = (0, \dots, 0, \frac{1}{2}R^2, \vartheta_0) \quad (41)$$

with $\vartheta_0 \in \frac{\pi}{k}\mathbb{Z}$, i.e. $w \in D$. Thus, we proved that the only $2k$ -cycle of $\phi_H \circ \phi_{\epsilon F}$ in the region $A(\frac{4}{9}, \frac{5}{9})$ is the one corresponding to the set D . Therefore it remains to prove that after possibly shrinking $\bar{\epsilon}$ there are no other fixed points of $(\phi_H \circ \phi_{\epsilon F})^{2k}$ outside $A(\frac{4}{9}, \frac{5}{9})$ except for $z = 0$. We argue by contradiction.

We assume that there exists a sequence $\epsilon_m \rightarrow 0$ and a sequence $(z^m)_{m \in \mathbb{N}}$ of points in $B^{2N}(R) \setminus A(\frac{4}{9}, \frac{5}{9})$ with

$$(\phi_H \circ \phi_{\epsilon_m F})^{2k}(z^m) = z^m \quad \forall m \in \mathbb{N}. \quad (42)$$

By Theorem of Arzelà-Ascoli we may assume that $z^m \rightarrow z^* \in B^{2N}(R) \setminus \text{int}A(\frac{4}{9}, \frac{5}{9})$ with

$$\phi_H^{2k}(z^*) = z^*. \quad (43)$$

It follows from Lemma 4 that $z^* = 0$ and thus for M sufficiently large

$$z^m \in B^{2N}(\frac{1}{3}R) \quad \forall m \geq M. \quad (44)$$

Then by definition of β the restriction of $\phi_{\epsilon_m F}$ to the ball $B^{2N}(\frac{1}{3}R)$ equals the identity. Moreover, since ϕ_H fixes all balls centered at zero we have

$$z^m = (\phi_H \circ \phi_{\epsilon_m F})^{2k}(z^m) = \phi_H^{2k}(z^m) \quad \forall m \geq M. \quad (45)$$

Applying again Lemma 4 we conclude that $z^m = 0$ for all $m \geq M$. This proves the Lemma. \square

Remark 6. Proposition 3 together with Lemma 5 implies that for all $0 < \epsilon < \epsilon_0$ the Hamiltonian diffeomorphism $\phi_H \circ \phi_{\epsilon F} : B^{2N}(R) \rightarrow B^{2N}(R)$ has no square root.

We are now in the position to prove Theorem 1.

Proof of Theorem 1. We choose $k \in \mathbb{Z}$, $\delta > 0$ and $0 < \epsilon < \epsilon_0$ (cp. Lemma 5) so that the Hamiltonian diffeomorphism

$$\phi_H \circ \phi_{\epsilon F} : B^{2N}(R) \rightarrow B^{2N}(R) \quad (46)$$

has precisely one $2k$ -cycle. By construction $\phi_H \circ \phi_{\epsilon F}$ equals a rotation with constant angular speed $2\pi\delta$ near the boundary of $B^{2N}(R)$. We extend the Hamiltonian function of $\phi_H \circ \phi_{\epsilon F}$ to $\tilde{H} : S^1 \times M \rightarrow \mathbb{R}$ which we can choose to be autonomous outside the Darboux ball B . If we choose $\delta > 0$ sufficiently small we can guarantee that outside B the only periodic orbits of \tilde{H} of period less or equal than $2k$ are critical points of \tilde{H} , see [HZ94, page 185]. In particular, $\phi_{\tilde{H}}$ has still precisely one $2k$ -cycle. Finally, by choosing k sufficiently large and δ and ϵ possibly

even smaller the C^∞ -norm of $\phi_H \circ \phi_{\epsilon F}$ and thus the C^∞ -norm of $\phi_{\tilde{H}}$ can be made arbitrarily small. Thus, with Proposition 3 the Theorem follows. \square

REFERENCES

- [HZ94] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 1994.
- [Mil84] J. Milnor, *Remarks on infinite-dimensional Lie groups*, Relativity, groups and topology, II (Les Houches, 1983), North-Holland, Amsterdam, 1984, pp. 1007–1057.
- [MS98] D. McDuff and D. A. Salamon, *Introduction to symplectic topology*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1998.

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